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The stability of a circular plate of shape memory alloy during a direct martensite transformation☆

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Abstract

In different formulations, analytical solutions are obtained of the problem of the axisymmetrical loss of stability of a circular plate of shape memory alloy undergoing a direct martensite transformation under a compression load. It is established that the solution based on the "fixed phase composition" concept gives an upper estimate of the critical load, while the solution based on the "continuing loading" concept gives a lower estimate. For a clamped plate, an intermediate solution is found using the "elastic unloading" concept that satisfies all the equations of the problem with zero variations in external loads and corresponds to a thickness of the additional phase transformation layer that is independent of the radial coordinate.

It has been found experimentally[1] that thermoelastic martensite transformations, occurring under compression stresses, may cause loss of stability of thin-walled elements of shape memory alloys (SMAs). Various concepts upon which a description of this effect may be based have been analysed:[2–5] the hypothesis of a "fixed phase composition", according to which, on transition to the neighbouring equilibrium state, the phase composition parameter does not change, and the concept of a "continuing phase transition", according to which the transition to the neighbouring equilibrium state is accompanied by an additional phase transformation. This last assumption is possible in two versions: the hypothesis of "continuing loading", according to which the active loads in the process of loss of stability can experience small perturbations, as a result of which all points of the cross-section undergo an additional phase transition, and the concept of "elastic unloading", in which a zone is formed on the convex surface of the plate during buckling that is free of an additional phase transition.

Within the framework of these concepts, solutions have been obtained of problems of the stability of a "Shanley column" on support rods of SMAs,[2,3] the stability of a rod[4] and the stability of a rectangular plate and strip plate of the same material.[5] It must be pointed out that the concepts of "elastic unloading" and "continuing loading" are not entirely alternative. Thus, for the case of a plate, a solution has been constructed[5] using the "elastic unloading" concept with a constant thickness of the additional phase transition layer. However, to justify this solution, the external loads acting on the plate must experience small perturbations, for which explicit expressions have been obained.

In the present paper, within the framework of different hypotheses, analytical solutions are obtained of the problem of the loss of stability of a continuous, circular, clamped SMA plate (without constraints on the radial displacement of

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the edges) or an SMA plate simply supported along its contour, caused by a direct martensite transformation occurring under a constant radial compression load, uniformly distributed over the plate contour.

1. Formulation of the problem

A continuous circular plate of constant thickness h and radius R, made of shape memory alloy (SMA) is considered. A polar coordinate system r, θ is introduced, the axes of which are positioned in the middle plane, while the origin is located at the centre of the plate. The plate is evenly loaded along its end edge with a constant, external, surface, compressive force p acting in the radial direction. The compression load is considered to be positive.

The surface force p is applied to the plate at such a high temperature that the stress state created does not cause a direct martensite transformation under isothermal conditions. The plate is then slowly cooled from the direct martensite transformation start temperature to the martensite transformation finish temperature. The temperature of all points of the plate at each instant of time is considered to be identical.

What we require is the minimum load for which, during a direct transformation, along with the trivial plane shape of the plate, distorted equilibrium shapes are also possible. The stability problem is solved in a linearized formulation within the framework of the theory of small strains and the kinematic Kirchhoff–Love hypotheses (with respect to total strains).

To solve the stability problem, we will use a simplified version of a previously proposed[6–8] system of constitutive relations for SMAs, which for a plane stress state (the axisymmetric problem) has the form

$$\varepsilon = \varepsilon^{(1)} + \varepsilon^{(2)} \tag{1.1}$$

$$\varepsilon_r^{(1)} = \frac{\sigma_r - \mu(q)\sigma_\theta}{E(q)}, \quad \varepsilon_\theta^{(1)} = \frac{\sigma_\theta - \mu(q)\sigma_r}{E(q)} \tag{1.2}$$

$$E(q) = \frac{q}{E_1} + \frac{1-q}{E_2}, \quad \frac{1}{G(q)} = \frac{q}{G_1} + \frac{1-q}{G_2}, \quad \mu(q) = \frac{E(q)}{2G(q)} - 1$$

$$d\varepsilon_r^{(2)} = \left(\frac{2\sigma_r - \sigma_\theta}{3\sigma_{(1)}} + a_0\varepsilon_r^{(2)}\right)dq, \quad d\varepsilon_\theta^{(2)} = \left(\frac{2\sigma_\theta - \sigma_r}{3\sigma_{(1)}} + a_0\varepsilon_\theta^{(2)}\right)dq \tag{1.3}$$

$$M_2 + k\sigma_i \le T \le M_1 + k\sigma_i, \quad kd\sigma_i - dT > 0 \tag{1.4}$$

$$q = \frac{1}{2}[1 - \cos(\pi(t + k^*\sigma_i))], \quad t = \frac{M_1 - T}{M_1 - M_2}, \quad k^* = \frac{k}{M_1 - M_2}, \quad \sigma_i = \sqrt{\sigma_r^2 + \sigma_\theta^2 - \sigma_r\sigma_\theta}. \tag{1.5}$$

where ε, $\varepsilon^{(1)}$ and $\varepsilon^{(2)}$ are the total, elastic and phase strains, σ_r, σ_θ and σ_i are the stress components and stress intensity, q is the volume fraction of the martensite phase, $E(q)$, E_1 and E_2 are the Young's modulus of the two-phase medium and its values in the martensite (subscript 1) and austenite (subscript 2) states, $G(q)$, G_1 and G_2 are the analogous values for the shear modulus, $\mu(q)$ is Poisson's ratio of the two-phase medium, T is the actual temperature, M_1 and M_2 are the martensite transformation start (subscript 1) and finish (subscript 2) temperatures in the stress-free material, and k, $\sigma_{(1)}$ and a_0 are constants of the material.

In accordance with Eqs. (1.1) and (1.3), when solving the problem, the volume effect of the reaction of the martensite transformation and the purely temperature strains are neglected, and from inequalities (1.4) and the final expression of system (1.5) it is clear that, in determining the conditions for the occurrence of direct transition (1.4), the influence of shear stresses is neglected.

2. Analysis of the subcritical state

It has been established[9] that, if the stress values obtained when solving a certain elastic problem are independent of the elasticity constants and coordinates of points of the body, then precisely the same stresses should be obtained when solving the problem of a direct martensite transformation of a body of SMA of the same shape and identical loading, provided that the temperature is independent of the coordinates of points of the body. Hence it follows that the subcritical stresses in the plate considered and the stress intensity σ_i during direct transformation will be given by the formulae

$$\sigma_r = \sigma_\varphi = -p, \quad \sigma_i = |p|. \tag{2.1}$$

The phase strains are found by integrating relations (1.3) taking into account equalities (2.1) and the zero initial conditions. We have

$$\varepsilon_r^{(2)} = \varepsilon_\varphi^{(2)} = -\frac{p}{3\sigma_{(1)}a_0}(\exp(a_0 q) - 1). \tag{2.2}$$

According to equalities (2.1), up to the instant when loss of stability occurs, there is uniform compression of the plate. The subcritical values of the deflection, angle of rotation and curvature are zero.

3. The system of equations for the perturbed state

It has been shown[4,5] that allowance for small temperature perturbations when analysing the stability of rods and plates of SMAs does not lead to any change in the values of the critical forces. Therefore, the temperature variation will be assumed to be zero in what follows. The straight normal hypothesis for the total strain is written for a plate, taking into account relations (1.1) and (1.2), in the form

$$\varepsilon_r^0 - x_3\kappa_r = \frac{\sigma_r - \mu(q)\sigma_\theta}{E(q)} + \varepsilon_r^{(2)}, \quad \varepsilon_\theta^0 - x_3\kappa_\theta = \frac{\sigma_\theta - \mu(q)\sigma_r}{E(q)} + \varepsilon_\theta^{(2)}. \tag{3.1}$$

where x_3 is the coordinate orthogonal to the plate, ε_r^0 and ε_θ^0 are the strains of the middle plane of the plate, and κ_r and κ_θ are the curvatures.

Taking variations of both sides of equalities (3.1) and using relations (1.3) and (2.2), we obtain

$$\delta\varepsilon_r^0 - x_3\delta\kappa_r = \frac{\delta\sigma_r - \mu\delta\sigma_\theta}{E} + g\delta q, \quad \delta\varepsilon_\theta^0 - x_3\delta\kappa_\theta = \frac{\delta\sigma_\theta - \mu\delta\sigma_r}{E} + g\delta q \tag{3.2}$$

$$g = g(p,q) = -p\left(\frac{1-\mu_1}{E_1} - \frac{1-\mu_2}{E_2} + \frac{1}{3\sigma_{(1)}}\exp(a_0 q)\right)$$
$$\delta q = \psi(q)k^*\delta\sigma_i U_+, \quad \psi(q) = \pi\sqrt{q(1-q)}, \quad \delta\sigma_i = -\frac{\delta\sigma_r + \delta\sigma_\theta}{2}, \quad U_+ = H(\delta\sigma_i), \tag{3.3}$$

where H is the Heaviside function, and μ_1 and μ_2 are Poisson's ratios for the martensite and austenite states. Here and below, all quantities without a variation sign relate to the subcritical state, and the argument q on the quantities E, G and μ without subscripts is omitted.

Solving Eqs. (3.2) and (3.3) for the stress variation, we obtain

$$\delta\sigma_r = \frac{E}{1-\mu^2}\frac{(\delta\varepsilon_r^0 - x_3\delta\kappa_r)(1+\xi^+U_+) + (\delta\varepsilon_\theta^0 - x_3\delta\kappa_\theta)(\mu - \xi^+U_+)}{1+\xi U_+}$$
$$\xi^+ = \xi^+(p,q) = -\frac{1}{2}Eg(p,q)\psi(q)k^*, \quad \xi = \xi(p,q) = \frac{2\xi^+(p,q)}{1-\mu}. \tag{3.4}$$

For $\delta\sigma_\theta$, a formula similar to (3.4) holds, in which the subscripts r and θ must change places.

To fix our ideas, suppose the lower part of the plate cross-section $-h/2 \le x_3 \le x_3^0$ on transition to the neighbouring equilibrium state experiences an additional phase transition (APT), and the upper part $x_3^0 \le x_3 \le h/2$ is deformed elastically. In the elastic part the differential condition for the occurrence of direct transformation (1.4) should be violated: $\delta\sigma_i \le 0$ (here, we have taken into account that $\delta T = 0$); in the APT zone, the opposite inequality should hold. Then, assuming a continuous variation of the stresses on passing through the boundary of the APT zone, it follows that, on this boundary itself, the relation $\delta\sigma_i = 0$ should hold. Substituting here the expression for the variation of the stress intensity (3.3), and using in the relation obtained the expression for the variations of the stress tensor components (3.4), written for the case $U_+ = 0$, we obtain an equation for determining the coordinate of the boundary of the APT zone:

$$\delta\varepsilon_r^0 + \delta\varepsilon_\theta^0 = x_3^0(\delta\kappa_r + \delta\kappa_\theta). \tag{3.5}$$

Substitution of expression (3.4), taking Eq. (3.5) into account, into the expressions for the variations of the forces and moments per unit length

$$\delta M_m = \int_{-h/2}^{h/2} \delta\sigma_m x_3 dx_3, \quad \delta N_m = \int_{-h/2}^{h/2} \delta\sigma_m dx_3,$$

where the subscript m takes the values of r, θ, yields

$$\begin{aligned} \delta M_r &= -D((\delta\kappa_r + \mu\delta\kappa_\theta) - \omega_1(\delta\kappa_r + \delta\kappa_\theta)) \\ \delta N_r &= D_1((\delta\varepsilon_r^0 + \mu\delta\varepsilon_\theta^0) - \omega_2(\delta\kappa_r + \delta\kappa_\theta)) \quad (r \leftrightarrow \theta) \end{aligned} \tag{3.6}$$

(the two unwritten relations are obtained by interchanging the subscripts r and θ). Here

$$\begin{aligned} D &= D(q) = \frac{Eh^3}{12(1-\mu^2)}, \quad D_1 = D_1(q) = \frac{Eh}{1-\mu^2} \\ \omega_1 &= \omega_1(p, z, q) = \frac{\omega}{4}(2-z)(1+z)^2, \quad \omega_2 = \omega_2(p, z, q) = \frac{h\omega}{8}(1+z)^2 \\ z &= \frac{2x_3^0}{h}, \quad \omega = \omega(p, q) = \frac{(1+\mu)\xi}{2(1+\xi)}. \end{aligned} \tag{3.7}$$

According to relations (3.6), the expressions for the variations of the forces and moments per unit length consist of an elastic part (the first terms in the round brackets) and an inelastic part (the second terms in the round brackets). The inelastic terms of the force and moment tensors are spherical tensors. Therefore, the differences $\delta M_r - \delta M_\theta$ and $\delta N_r - \delta N_\theta$ are calculated using the usual elastic relations with variable moduli depending on the parameter q. At the same time, the sums of the variations of the moments or forces are proportional respectively to the sums of the variations of the curvatures or strains of the middle surface [to make sure of this for the case of forces, it is necessary to use relation (3.5)]. These considerations facilitate the derivation of the resolving equation of the problem.

To relations (3.6) it is necessary to add the equilibrium equations

$$r\delta M_r' + \delta M_r - \delta M_\theta + N_r r\delta\varphi = 0, \quad \delta N_r' + \frac{\delta N_r - \delta N_\theta}{r} = \delta Q_r, \tag{3.8}$$

where δQ_r is the variation of the external surface load, and the equation of compatibility

$$\delta\varepsilon_r - \delta\varepsilon_\theta - r\delta\varepsilon_\theta' = 0,$$

where the prime denotes a derivative with respect to the coordinate r. The latter equation, taking into account the left-hand sides of representations (3.1), and also linear expressions linking the curvatures and the angle of rotation

$$\delta\kappa_r = \delta\varphi', \quad \delta\kappa_\theta = \delta\varphi/r, \tag{3.9}$$

is equivalent to a like equation but written only for variations of the strains in the middle plane:

$$\delta\varepsilon_r^0 - \delta\varepsilon_\theta^0 - r\delta\varepsilon_\theta^{0\prime} = 0. \tag{3.10}$$

Instead of the condition of compatibility (3.10), it is possible to use expressions for the variations in the corresponding strains in terms of the variation of the radial displacement

$$\delta\varepsilon_r^0 = \delta u', \quad \delta\varepsilon_\theta^0 = \delta u/r. \tag{3.11}$$

As a result, all the unknown functions of the problem can be expressed in terms of the two quantities, du and dφ, for which two differential equations must be satisfied (each of the second order), obtained by substituting expressions (3.5), (3.6), (3.9) and (3.11) into Eq. (3.8).

The boundary conditions at the centre of the plate (at $r=0$) are the conditions of zero variation of the angle of rotation and zero variation of the radial displacement

$$\delta u(0) = 0, \quad \delta\varphi(0) = 0. \tag{3.12}$$

The boundary conditions on the outer contour of the plate for the angle of rotation depend on the type of fastening the plate. For the clamping condition of zero variation of the angle of rotation

$$\delta\varphi(R) = 0. \tag{3.13}$$

For simple support there is the condition of zero variation of the radial bending moment (3.6)

$$R\delta\varphi'(R)(1-\omega_1)+\delta\varphi(R)(\mu-\omega_1) = 0. \tag{3.14}$$

The boundary condition on the outer contour for the variation of the radial displacement reduces, generally speaking, to specifying the variation of the external load on this contour: $\delta N_r = -h\delta p$, or

$$\delta u'(R)\left(1-\frac{2\omega_2}{hz}\right)+\frac{\delta u(R)}{R}\left(\mu-\frac{2\omega_2}{hz}\right) = -\frac{h\delta p}{D_1}. \tag{3.15}$$

Suppose the problem is solved when there are no variations of the external loads (i.e. using the "elastic unloading" concept). In this case, in the second equation of system (3.8) and in Eq. (3.15), it is necessary to assume that $\delta Q_r=0$ and $\delta p=0$. The system of equations obtained with the corresponding boundary conditions is sufficient to solve the stability problem.

With the "continuing loading" concept, the stability problem for SMAs, generally speaking, becomes indeterminate, since the perturbations of the external loads are unknown. However, it is possible to use a formulation of the problem in which, in a certain class of solutions, that solution is sought for which the critical values of the external loads are a minimum, provided that the perturbations of the external loads are small.

4. Solutions for the case of a constant thickness of the zone of additional phase transition

In the present paper, as the class of solutions mentioned above, we will consider those solutions for which the quantity x_3^0 (or z) is independent of the radial coordinate r (i.e. the thickness of the APT zone is identical at all points of the plate). Combining the expressions for δN_r and δN_θ (3.6), taking Eq. (3.5) into account, we obtain

$$\delta N_r+\delta N_\theta = \lambda(\delta\varepsilon_r^0+\delta\varepsilon_\theta^0), \quad \lambda = \lambda(p,z,q) = \frac{Eh}{1-\mu}\left(1-\frac{1}{4}\frac{\xi}{1+\xi}\frac{(1+z)^2}{z}\right). \tag{4.1}$$

The quantity q is assumed to be independent of the coordinate r, and therefore the q-dependent functions $E(q)\mu(q)$ and $\xi(p,q)$ in expression (4.1) for λ are independent of r. Thus, according to equality (4.1), the quantity z is independent of r in the case when the sum of the variations of the membrane forces is proportional to the sum of variations of the strains of the middle surface, with a proportionality factor that does not depend on r.

Substituting into relation (3.5) the expressions for the variations of the curvatures (3.9) and strains (3.11) in terms of the variations of the angle of rotation and radial displacement, we obtain $(r\delta u)' = x_3^0(r\delta\varphi)'$. Integration with respect to r under conditions where x_3^0 is independent of the radius, yields $(r\delta u) = x_3^0 r\delta\varphi + C$. From this equality and the condition that δu and $\delta\varphi$ are bounded as $r\to 0$, it is possible to obtain $C=0$, from which it follows that

$$\delta u = zh\delta\varphi/2. \tag{4.2}$$

Thus, for the class of solutions considered, the variations $\delta\varphi$ and δu are proportional, with a factor that is independent of the coordinate r. Consequently, any homogeneous linear equation or boundary condition that is satisfied for one of these variations must also be satisfied for the other. Therefore, on the basis of the "elastic unloading" concept with zero variations of the external forces, each of the functions $\delta\varphi$ and δu should satisfy two differential equations and two sets of boundary conditions. It is clear that such a situation can only occur when a special selection is made of the functions in the system of equations.

When the parameter z is independent of r, substitution of the expressions for the variations of the moments (3.6) into the first equilibrium Eq. (3.8), taking into account expressions (3.9) for the variations of the curvatures, leads to the equation

$$\delta\varphi'' + \frac{\delta\varphi'}{r} - \left(\frac{1}{r^2} - \alpha^2\right)\delta\varphi = 0, \quad \alpha^2 = \alpha^2(p, z, q) = \frac{hp}{D(q)(1-\omega_1(p, z, q))}, \tag{4.3}$$

which differs solely in the value of the coefficient of $\delta\varphi$ from the equation of stability for an elastic circular plate,[10,11] [the latter is obtained from Eq. (4.3) if it is assumed that $\omega_1 = 0$ and the cylindrical stiffness $D(q)$ is considered to be constant]. Therefore, the procedure for solving Eq. (4.3) does not differ from the procedure for investigating the corresponding equation for elastic plates.[10,11]. The difference is that, for a plate of SMA, unlike for an elastic plate, the required eigenvalue depends both on q and on z, and here, besides Eq. (4.3), it is necessary for the second equation of system (3.8), with the corresponding boundary condition (3.15), to be satisfied, and also relation (3.5). When z is independent of r, this system can be solved successively. Initially, from Eq. (4.3), the eigenvalue and the eigenfunction $\delta\varphi$, which depend on the quantities z and q, are determined. Then, changing, by means of relation (4.2), to the variation δu, and substituting the solution obtained into the second equation of system (3.8), it is possible to find the value of z. A version is possible with a specification from some considerations of the quantity z and subsequent determination of the corresponding values of the load variations on the basis of the second equation of (3.8) and condition (3.15). As such considerations it is possible, in particular, to use the requirements that the critical loads should be a minimum or a maximum with respect to z. In the elastic case, the system of two equations of stability splits, and for the second equation of (3.8) a trivial solution is possible.

The replacement of the independent variable r by $\upsilon = r\alpha$ enables us to reduce Eq. (4.3) to a Bessel equation, the solution of which is written in terms of Bessel functions of the first and second kind with an index of unity in the form $\varphi(v) = C_1 J_1(v) + C_2 Y_1(v)$ For a continuous circular plate, from the second boundary condition of system (3.12) it follows that $C_2 = 0$, i.e.

$$\varphi(\alpha r) = C_1 J_1(\alpha r), \quad \delta u(\alpha r) = C_1 hz J_1(\alpha r)/2. \tag{4.4}$$

The second formula of system (4.4) was obtained from the first by means of relation (4.2).

For a plate simply supported along its contour, from the boundary condition (3.14) we obtain the equation

$$J_0(U)(1-\omega_1(p, z, q)) - U^{-1} J_1(U)(1-\mu(q)) = 0, \quad U = \alpha R, \tag{4.5}$$

where J_0 is a Bessel function of the first kind with zero index. The smallest root of this equation U_1 depends, generally speaking, on p, z and q. For a plate clamped along its contour, boundary condition (3.13) yields $J_1(U) = 0$. Its smallest root U_2 is roughly 3.8317.

Taking into account the equality $\alpha R = U_i$, from the second formula of system (4.3) we obtain the required equations for the critical load

$$p = D(q) U_i^2 (1-\omega_1(p, z, q))/(hR^2). \tag{4.6}$$

Here and below, $i = 1$ corresponds to simple support, and $i = 2$ corresponds to clamping. For the case of simple support it is more convenient in general to solve not system (4.5), (4.6) but the equation obtained directly by substituting the expression $U = \alpha R$ into the first formula of (4.5). As a result, taking into account the dependence of α on p, z and q (4.3), Eq. (4.5) enables us to determine the critical load p as a function of z and q.

Minimization of the functions $p = p(z, q)$ obtained for the cases of clamping and simple support with respect to q, and, possibly, depending on the statement of the problem, with respect to z, enables us to find the critical value of the load p^*.

According to formula (4.6), taking into account relations (3.7), it can be shown that the critical load p as a function of z in the segment $[-1, 1]$ for fixed q has its greatest and least values at the points $z = -1$ and $z = 1$ respectively.

An important feature of the solution of the problem of the stability of a plate of SMA clamped along its contour must be pointed out. The fact is that, in accordance with solution (4.4), the variation of the doubled mean curvature $\delta\kappa_r + \delta\kappa_\theta$ vanishes at a certain value of $r = r_0 < R$, and, on passing through this point, it changes sign. When r passes through the value r_0, that surface of the plate to which the region of elastic unloading with $\delta\sigma_i < 0$ is adjacent will now border the region of additional loading with $\delta\sigma_i > 0$. As a result, according to Eq. (3.5), the sign of x_3^0 and z should change. Consequently, in this case it is a matter not of a constant value of z (independent of r) but of a coordinate that, without changing in absolute magnitude, can change sign (i.e. is a discontinuous function). Nonetheless, it is possible to show that the solution obtained satisfies all the equations of the problem both to the left and to the right of the point of discontinuity (the final system will differ slightly from that given above since it is formulated for conditions where the region $-h \le x_3 \le x_3^0$ is the zone of unloading while the region $x_3^0 \le x_3 \le h$ is the APT zone).

In the case where the function $z = z(p, q)$ or $\lambda = \lambda(p, q)$ is assigned, the magnitude of the variation of the external load necessary for all the equations of the stability problem to be satisfied can be found by substituting the expressions

$$\delta N_r - \delta N_\theta = D_1(1-\mu)(\delta\varepsilon_r^0 - \varepsilon_\theta^0)$$

$$\delta N_r' = D_1\left(\left(\frac{hz}{2} - \omega_2\right)(\delta\kappa_r + \delta\kappa_\theta)' - (1-\mu)\frac{\delta\varepsilon_r^0 - \delta\varepsilon_\theta^0}{r}\right),$$

obtained from the second equation of (3.6), taking into account relations (3.5) and (3.10), into the second equation of system (3.8). We obtain

$$\delta Q_r = D_1\left(\frac{hz}{2} - \omega_2\right)(\delta\kappa_r + \delta\kappa_\theta)'. \tag{4.7}$$

The substitution into this equation of expressions (3.9), in which the function φ is represented in the form (4.4), and also the use of the expression for λ (4.1), enables us finally to obtain

$$\delta Q_r = -\frac{C_1 G h^2 z}{2}\left(1 + \frac{\lambda}{2Gh}\right)\left(\frac{U_1}{R}\right)^2 J_1\left(\frac{U_1 r}{R}\right). \tag{4.8}$$

The magnitude of δp can be found from condition (3.15) using the second formula of system (4.4).

If the concept of "fixed phase composition" (CFPC)[3,4] is adopted, then, on transition to the neighbouring equilibrium state, APT does not occur. The values of the variations of the forces and moments, and also the magnitude of the critical force, corresponding to the solution of the problem in such a formulation, can be obtained formally by assuming in the formulae obtained above that $z = -1$. As a result, according to relations (3.7), $\omega_1 = 0$ and $\omega_2 = 0$ will be obtained, in the formula for the variations of the forces and moments (3.6) only "elastic" terms will remain, and formula (4.6) for the critical force will be transformed into a dependence characteristic of an elastic circular plate,[10,11] in which account is taken of the variability of the cylindrical stiffness $D(q)$ on phase transition:

$$p = \frac{D(q)U_i^2}{hR^2}. \tag{4.9}$$

According to relations (4.1), in this solution $\lambda = Eh/(1-\mu)$, and from equality (4.8) the following equation is obtained

$$\delta Q_r = \frac{C_1 G h^2}{1-\mu}\left(\frac{U_i}{R}\right)^2 J_1\left(\frac{U_i r}{R}\right). \tag{4.10}$$

Therefore, transition to the neighbouring equilibrium state without APT can occur within the framework of system of constitutive relations (1.1)–(1.5) only when there are perturbations of the external load. It can be shown that, owing to variations of the external surface forces (4.10) and the end load δp, an additional stress state occurs in the plate, the variation of the intensity of which is negative at each point of the cross-section and zero on its lower boundary.

It should be noted that the same expression for the critical force (4.9) can be arrived at by solving the stability problem within the framework of system (1.1)–(1.5), provided that the material parameter k is zero, i.e. according to the terminology adopted earlier[12] when solving the problem in an uncoupled statement. In this last case, on transition to the neighbouring equilibrium state, APT cannot in principle occur, and relation (3.5) loses its meaning and does not occur in the system of equations of the problem. The existence of such a solution does not require variations of the external load of type (4.7), (4.8) since the equations for the strain of the middle plane and the corresponding forces have a zero solution admissible if relation (3.5) does not occur in the system of equations. Thus, solutions of the stability problem using the CFPC and in an uncoupled statement, although essentially different, lead to the same value of the critical force (4.9), which is the maximum of all possible values obtained when solving the problem, provided the quantity z is independent of r.

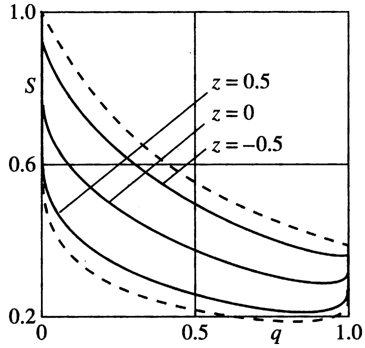

Fig. 1.

Below, we will consider the alternative case of the lowest possible critical load, obtained when $z=1$. This corresponds to the concept of "continuing loading" (CCL),[3,4] according to which on transition to the neighbouring equilibrium state all the points of the cross-section experience APT. In this case, according to relations (3.7), $\omega_1(p, z, q)=\omega(p, q)$ and the critical load depends only on the phase composition parameter:

$$p = \frac{D(q)U_i^2}{hR^2}(1-\omega(p,q)).$$

This solution is consistent with the equilibrium equation for membrane forces (3.8) only when there is a perturbation of the surface load δQ_r, determined using relation (4.7) with $z=1$, $\omega_2(q, z)=h\omega(q)/2$. It can be shown that, for the case of simple support, within the framework of the given solution the boundary condition (3.15) is satisfied with zero variations of the end load: $\delta p=0$. In fact, from relations (3.7) it is easy to establish that, with $z=1$, the equality $\omega_1=2\omega_2/(hz)$ is satisfied. Then, taking relation (4.2) into account, from condition (3.14) it follows that the left-hand side of relation (3.15) is zero, from which the assertion is obtained.

The values of the critical forces obtained for intermediate values of $z\in(-1, 1)$ lie between solutions based on the CCL and the CFPC. This fact is illustrated in Fig. 1 for the case of the clamped plate with a relative value of its radius $R/h=20$; the dependence on q of the dimensionless critical forces S, related to critical load $p = U_1^2D(0)/(hr^2)$, with isothermal loss of stability of an elastic plate with austenite values of the moduli is given for the CCL (the lower dashed curve), for different values of z and for the CFPC (the upper dashed curve). This and all subsequent calculations were carried out for the following values of parameters of the material, characteristic of titanium nickelide

$$\frac{E_1}{E_2} = \frac{1}{3}, \quad \mu_1 = 0.48, \quad \mu_2 = 0.33, \quad k^*E_2 = 480, \quad \frac{\sigma_{(1)}}{E_2} = 0.049, \quad a_0 = 0.718.$$

The critical load corresponding to loss of stability p^* for each value of z can be determined from these curves as the ordinate of the minimum point. It is interesting to note that only for the upper curve obtained with the CFPC does the lowest value of critical load correspond to $q=1$, i.e. loss of stability occurs in the entirely martensite phase state, corresponding to the minimum values of the moduli of elasticity. For $-1<z\leq 1$, the minimum on the curves $p=p(q)$ is reached for values $0.5<q^*<1$. The position of the minimum is defined on the one hand by the point $q=0.5$, where the rate of change in the phase composition parameter is maximum, and on the other hand by the point $q=1$, where the moduli of elasticity are a minimum. Therefore, in the case of failure to account for APT, the minimum value of the critical load corresponds to the point $q=1$ (the upper curve). If, however, in the formulae obtained above it is assumed that $E_1=E_2$ and $\mu_1=\mu_2$, i.e. the variability of the moduli of elasticity is ignored, then all the curves for $-1<z\leq 1$ will have a minimum at $q=0.5$. It should be noted that, according to experimental data,[1] loss of stability during a direct transformation never occurs when $q=1$ but is always observed at some intermediate point of the phase transition.

5. The exact solution of the stability problem when there are no perturbations of the external load

In accordance with relation (4.7) or (4.8) a solution of the stability problem exists in which the coordinate of the boundary of the APT zone is independent of r, and in this case the equilibrium equation for membrane forces is satisfied with zero right-hand sides, i.e. for zero variations of the external loads. For this solution, the following relations must be satisfied

$$\omega_2(q, z) = hz/2, \quad \lambda = -2G(q)h, \tag{5.1}$$

which ensure that the quantity δQ_r is zero. It can be shown that, for a clamped plate, when relations (5.1) are satisfied, the variation of the end compression force δp will also be zero. In fact, for $\delta p = -\delta N_r(R)/h$, on the basis of the second relation of system (3.6), taking into account equalities (3.5), (5.1) and (4.2), it is possible to obtain

$$\delta p = D_1(1-\mu)\delta\varphi(R)/(hR).$$

From this it follows that, for a clampled plate, by virtue of the boundary condition (3.13), $\delta p = 0$. It should be noted that, for a simply supported plate, when condition (5.1) is satisfied, the values of δQ_r will be zero but the values of

$$\delta p = C_1 D_1(1-\mu)\delta\varphi(R)J_1(\alpha R)/(hR).$$

will be non-zero.

From relations (5.1), taking the expression for ω_2 (3.7) into account, it is possible to obtain the equation

$$z^2 + 2(1-2/\omega)z + 1 = 0. \tag{5.2}$$

It can be shown that, for values of the constants of elasticity that are characteristic of SMAs of the titanium nickelide type ($E_2 \geq E_1$, $\mu_2 \leq \mu_1$), the condition $0 < \omega \leq 1$ is satisfied, and Eq. (5.2) has two real roots, one of which exceeds unity, while the other

$$z = \frac{1-\sqrt{1-\omega}}{1+\sqrt{1-\omega}} \in [0, 1] \tag{5.3}$$

for all values of $q \in [0,1]$. Expression (5.3) serves as the required unique solution of the problem in question for z. It must be noted that, unlike the solutions obtained in the previous section, in this case the parameter z is not constant but necessarily depends on q. From the non-negativity of z [Eq. (5.3)] it follows that, within the framework of the solution obtained, the APT zone $[-1, z]$ should occupy a large part of the plate cross-section.

Substitution of expression (5.3) into the expression for ω_1 (3.7) and the result of this substitution into expressions (4.6) for the critical force yields

$$p = \frac{D(q)U_i^2}{hR^2}\frac{4(1-\omega)}{2-\omega+2\sqrt{1-\omega}}. \tag{5.4}$$

The solution of system of Eqs. (5.3), (5.4) yields the function $z = z(q)$, the graphs of which for the case of clamping are presented in Fig. 2 for different values of R/h. It can be seen that for fairly thick plates these graphs comprise, in any case, curves that are extremely flat in their central section, reaching maximum values of z_1 at certain mid-points $q = q_1$ of the interval (0, 1).

The graph of the dimensionless critical force S against q, obtained by solving the same system for $R/h = 20$, is given in Fig. 3 (the solid curve); the dashed curve represents the dependence of S on q, found by solving problem (4.6) for $z = z_1 = \text{const}$. According to the data in Fig. 3, the abscissa of the minimum point of the limit load with respect to q does

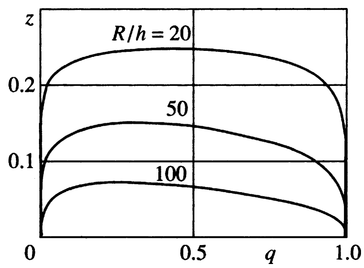

Fig. 2.

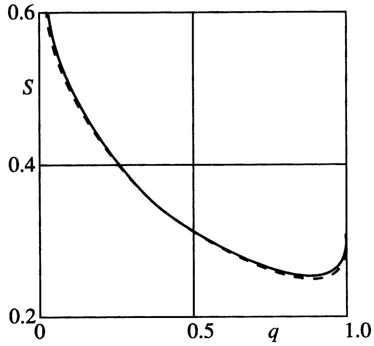

Fig. 3.

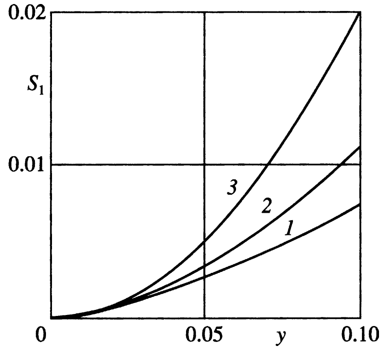

Fig. 4.

not coincide with the abscissa of the maximum point of z with respect to q. The limit loads p^* themselves, determined from the solution for variable z and obtained for a constant maximum value $z = z_1$, hardly differ.

Figure 4 shows graphs of the dependences of the dimensionless critical load $S_1 = p^*h^3/(D(1))U_2^2)$, related to the critical load of isothermal loss of stability in the martensite state, against the relative thickness of the plate $y = h/R$ for the case of clamping. Curve *1* was obtained using the concept of "continuing loading", curve *2* was obtained using the concept of "elastic unloading" with zero variations in the external forces, and curve *3* was obtained using the concept of "fixed phase composition". As can be seen, the difference between these solutions is small for comparatively thin plates and increases sharply as their thickness increases. The critical load for fairly thick plates may be many times less than the critical force corresponding to isothermal loss of stability in the martensite phase state, which is consistent with experimental data.[1]

In Fig. 5, the dependences of the relative values of the limit loads S on q for the cases of clamping (the solid curves) and simple support (the dashed lines) are compared. All loads are related to the critical force corresponding to

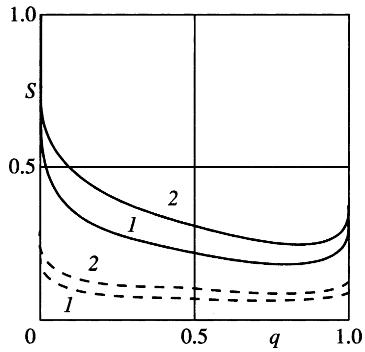

Fig. 5.

isothermal loss of stability of a clamped plate in the austenite phase state. Curves *1* were obtained using the "continuing loading" concept, and curves *2* were obtained using the "elastic unloading" concept. According to Fig. 5, the difference between the minimum critical values of the loads with respect to q, determined within the framework of these concepts, is considerably higher for the case of clamping than for simple support.

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